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A note on eigenvalue asymptotics for Hill's equation

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ABSTRACT

In this note, we give asymptotic estimates for the periodic and anti-periodic eigenvalues of Hill's equation up to an order of $o(n^{-1})$. We also apply the estimates to give a short proof of the associated Ambarzumyan Theorem.

1. Introduction

Consider Hill's equation:

$$-y'' + q(x)y = \lambda y, \tag{1.1}$$

under the periodic boundary conditions $y(0) = y(1),\ y'(0) = y'(1)$, or the anti-periodic boundary conditions $y(0) = -y(1),\ y'(0) = -y'(1)$. Here we assume that $q \in L^1(0, 1)$ is a real-valued function extended to $\mathbb{R}$ by periodicity. Denote by $\{\mu_k\}_{k\geq 1}$, $\{\nu_k\}_{k\geq 0}$, $\{\lambda_n\}_{n\geq 0}$ and $\{\hat{\lambda}_n\}_{n\geq 1}$ the Dirichlet, Neumann, periodic and anti-periodic eigenvalues of (1.1), respectively. It is well known [1,2] that

(a)

$$\lambda_0 < \hat{\lambda}_1 \leq \hat{\lambda}_2 < \lambda_1 \leq \lambda_2 < \cdots \hat{\lambda}_{2n-1} \leq \hat{\lambda}_{2n} < \lambda_{2n-1} \leq \lambda_{2n} < \cdots \to \infty.$$

The intervals $(\hat{\lambda}_{2n-1}, \hat{\lambda}_{2n})$ and $(\lambda_{2n-1}, \lambda_{2n})$ are called the $(2n-1)$-th and $2n$-th instability intervals respectively.

(b) $\nu_0 \leq \lambda_0$ and

$$\hat{\lambda}_{2n-1} \leq \nu_{2n-1}, \qquad \mu_{2n-1} \leq \hat{\lambda}_{2n},$$
$$\lambda_{2n-1} \leq \nu_{2n}, \qquad \mu_{2n} \leq \lambda_{2n}.$$

In this note, we shall give asymptotic estimates for the periodic eigenvalues λ_n's and the anti-periodic eigenvalues $\hat{\lambda}_n$'s of Hill's equation up to $o(n^{-1})$. One of our results is

Theorem 1.1. *For $m = 2n$ or $2n - 1$,*

$$\lambda_m = (2n\pi)^2 + 2\int_0^1 q(x)\mathrm{d}x + 2(-1)^m[A_{4n}^2 + B_{4n}^2]^{\frac{1}{2}} + o\left(\frac{1}{n}\right), \tag{1.2}$$

$$\hat{\lambda}_m = (2n-1)^2\pi^2 + 2\int_0^1 q(x)\mathrm{d}x + 2(-1)^m[A_{4n-2}^2 + B_{4n-2}^2]^{\frac{1}{2}} + o\left(\frac{1}{n}\right), \tag{1.3}$$

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where A_n and B_n be the Fourier coefficients of q. That is,

$$A_n = \frac{1}{2}\int_0^1 \cos(n\pi t)q(t)\mathrm{d}t, \qquad B_n = \frac{1}{2}\int_0^1 \sin(n\pi t)q(t)\mathrm{d}t. \tag{1.4}$$

In [3], Coskun and Harris gave a similar asymptotic estimate, with the error term $O(\eta^2(n)) + O(n^{-1}\eta(n))$, where

$$\eta(n) := \eta_n = \sup\left\{\frac{1}{A}\left|\int_x^{\tau+1} \exp\{4n\pi \mathrm{i}t\}q(t)\mathrm{d}t\right| : \tau \le x \le \tau+1, 0 \le \tau \le 1\right\},$$

with $A = \int_x^{\tau+1} |q(t)|\,\mathrm{d}t$. Note that $\eta(n) \to 0$ as $n \to \infty$. This note provides a straightforward eigenvalue estimates with a higher order of accuracy. Furthermore, we apply this asymptotic estimates to give a short proof of an Ambarzumyan Theorem for periodic and anti-periodic Sturm–Liouville problems. A more general theorem for vectorial Sturm–Liouville equation, using the theory of transformation operator, was recently established by Yang–Huang–Yang [4].

Theorem 1.2. (a) *If all periodic eigenvalues of Hill's equation (1.1) are nonnegative and they include $\{(2n\pi)^2 : n \in \mathbb{N}\}$, then $q = 0$ a.e.*

(b) *If all anti-periodic eigenvalues of Hill's equation (1.1) are not less than π^2 and they include $\{(2n-1)^2\pi^2 : n \in \mathbb{N}\}$, and $\int_0^1 q(x)\cos(2\pi x)\mathrm{d}x \ge 0$, then $q = 0$ in a.e.*

Note that in the above statement, we do not need to assume that each of the eigenvalues $\{(2n\pi)^2 : n \in \mathbb{N}\}$ is of multiplicity 2 as in [4, Theorem 1.1]. Furthermore it is now well known that an subset of the spectra plus the least eigenvalue will be sufficient to obtain the Ambarzumyan Theorem [5] so that the above condition can be even slightly weakened.

The idea we use to derive the eigenvalue estimates in Theorem 1.1 comes from Hochstadt [6]. In 1965, Hochstadt showed that if all finite instability intervals vanish, then $q = 0$ a.e.. He also showed that if $\{\mu_n(\tau)\}_{n\ge 1}$ are the eigenvalues of

$$\begin{aligned} &-y'' + q(x+\tau)y = \lambda y, \\ &y(0) = y(1) = 0, \end{aligned} \tag{1.5}$$

then $\hat{\lambda}_1 \le \mu_1(\tau) \le \hat{\lambda}_2,\ \lambda_1 \le \mu_2(\tau) \le \lambda_2$, etc., and

$$\lambda_{2n-1} = \min_\tau \mu_{2n}(\tau), \qquad \lambda_{2n} = \max_\tau \mu_{2n}(\tau), \tag{1.6}$$

$$\hat{\lambda}_{2n-1} = \min_\tau \mu_{2n-1}(\tau), \qquad \hat{\lambda}_{2n} = \max_\tau \mu_{2n-1}(\tau). \tag{1.7}$$

Let λ_m ($m = 2n$ or $2n-1$) be a periodic eigenvalue associated with the eigenfunction y_m of (1.1) with $2n$ zeros in $[0, 1)$. If $x_1^{(m)}$ is the first zero of y_m in $[0, 1)$, then $u_m(t) = y_m(t + x_1^{(m)})$ gives the $2n$-th Dirichlet eigenfunction associated with λ_m for the translated equation

$$-u''(t) + q(t + x_1^{(m)})u(t) = \lambda u(t).$$

That is, if λ_m is a periodic eigenvalue for $q(t)$, then it is the $2n$-th Dirichlet eigenvalue for the translated potential $q(\cdot + x_1^{(m)})$. This is an easy part of Hochstadt's result (1.6). The novelty of (1.6) is that all the other translated potential $q(\cdot + x_1^{(m)})$, where τ is not a zero of y_m, gives a Dirichlet eigenvalue $\mu_m(\tau) \le \lambda_m$. (In these cases, it is clear that $u_m'(\tau) \ne u_m'(1+\tau)$.) Thus (1.6) states that λ_m coincides with the Dirichlet eigenvalue $\mu_m(x_1^{(m)})$, which is extremal among all $\mu_m(\tau)$, $\tau \in [0, 1)$.

Using the well-known result [7, Lemma 2.2(a)], we obtain the Dirichlet eigenvalue $\mu_n(\tau)$ and hence the estimate of periodic eigenvalue is derived by (1.6).

2. Proof of main results

Proof of Theorem 1.1. First of all, for any fixed $\tau \in [0, 1)$, let $Q(t) = q(t+\tau)$ and consider the Dirichlet problem

$$\begin{cases} -Y''(t) + Q(t)Y(t) = \lambda Y(t), \\ Y(0) = Y(1) = 0. \end{cases} \tag{2.1}$$

As it is known that the asymptotic expansion of eigenvalues μ_n for the Dirichlet problem (2.1) is given [7, Lemma 2.2(a)] by

$$\sqrt{\mu_n} = n\pi + \frac{1}{2n\pi}\int_0^1 (1 - \cos 2n\pi t)Q(t)\mathrm{d}t + o\left(\frac{1}{n^2}\right),$$

where the error term $o(\frac{1}{n^2})$ is only dependent on $\|Q\|_2 = \|q\|_2$ and so is uniform in τ. Now let $s_{2n} := \sqrt{\mu_{2n}(\tau)}$. Thus

$$s_{2n} = 2n\pi + \frac{1}{4n\pi}\int_0^1 (1 - \cos 4n\pi t)q(t+\tau)\mathrm{d}t + o\left(\frac{1}{n^2}\right). \tag{2.2}$$

We then derive the second term as follows. Since q is periodic, we have $\int_0^1 q(t+\tau)\mathrm{d}t = \int_0^1 q(t)\mathrm{d}t$ and

$$\begin{aligned}\int_0^1 -\cos(4n\pi t)q(t+\tau)\mathrm{d}t &= -\int_\tau^{1+\tau} \cos(4n\pi(t-\tau))q(t)\mathrm{d}t,\\ &= -\cos(4n\pi\tau)\int_0^1 \cos(4n\pi t)q(t)\mathrm{d}t - \sin(4n\pi\tau)\int_0^1 \sin(4n\pi t)q(t)\mathrm{d}t,\\ &= -2[A_{4n}^2 + B_{4n}^2]^{\frac{1}{2}}\sin(4n\pi\tau+\phi)\end{aligned}$$

where A_{4n} and B_{4n} were defined in (1.4) and $\phi = \tan^{-1}(A_{4n}/B_{4n})$. Hence

$$\begin{aligned}\sqrt{\lambda_{2n}} &= \max\left\{\sqrt{\mu_{2n}(\tau)} : \tau \in [0,1)\right\},\\ &= 2n\pi + \frac{1}{2n\pi}\int_0^1 q(t)\mathrm{d}t + \frac{1}{2n\pi}[A_{4n}^2 + B_{4n}^2]^{\frac{1}{2}} + o\left(\frac{1}{n^2}\right).\end{aligned}$$

The maximum is achieved when $\tau = \frac{\frac{3}{2}\pi-\phi}{4n\pi}$. Similarly, for the odd periodic eigenvalues,

$$\begin{aligned}\sqrt{\lambda_{2n-1}} &= \min\left\{\sqrt{\mu_{2n}(\tau)} : \tau \in [0,1)\right\},\\ &= 2n\pi + \frac{1}{2n\pi}\int_0^1 q(t)\mathrm{d}t - \frac{1}{2n\pi}[A_{4n}^2 + B_{4n}^2]^{\frac{1}{2}} + o\left(\frac{1}{n^2}\right).\end{aligned}$$

The estimates (1.3) for anti-periodic eigenvalues can be derived analogously, with the help of the following estimate

$$\hat{s}_m = (2n-1)\pi + \frac{1}{2(2n-1)\pi}\int_0^1 (1-\cos(2(2n-1)\pi t))q(t+\tau)\mathrm{d}t + o\left(\frac{1}{n^2}\right), \tag{2.3}$$

noting that with $\hat{s}_m := \hat{\mu}_m(\tau)$ where $\hat{\mu}_m$ is the mth eigenvalue for the Dirichlet problem with an odd number of interior zeros. □

Proof of Theorem 1.2. (a) When n is large, the eigenvalue $(2n\pi)^2$ has to be the $2n$-th or $(2n-1)$-th eigenvalue. In either case, by (1.2),

$$\int_0^1 q(x)\mathrm{d}x = 0.$$

Then as in [4], we may use the Rayleigh–Ritz inequality with $u \equiv 1$ as a test function

$$0 \le \lambda_0 \le \frac{\int_0^1 (u'^2 + qu^2)\mathrm{d}x}{\int_0^1 u^2\,\mathrm{d}x} = \int_0^1 q(t)\mathrm{d}t = 0.$$

This implies 0 is the first eigenvalue corresponding to the eigenfunction 1 and so $q = 0$.

(b) Let $\hat{s}_m := \sqrt{\hat{\lambda}_m}$. From (1.3), $\int_0^1 q = 0$. Then by assumption and the above Rayleigh–Ritz inequality, one can easily show that $\sin(\pi x)$ is an eigenfunction associated with the least eigenvalue $\hat{\lambda}_1 = \pi^2$. Therefore $q = 0$. □

We shall remark that Theorem 1.2(a) can also be proved with Yurko's argument [8]. Since u_0 does not vanish in $(0,1)$,

$$0 = \int_0^1 q = \int_0^1 \frac{u_0'' + \lambda_0 u_0}{u_0} = \int_0^1 \left(\frac{u_0'}{u_0}\right)^2 + \lambda_0.$$

Since $\lambda_0 \ge 0$ by assumption, it forces $\lambda_0 = 0$ and u_0 to be constant.

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